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LETTER TO THE EDITOR

Relative positions of limit cycles in a Kolmogorov-type system

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Abstract. To estimate the relative position of limit cycles for a system is always useful in the qualitive analysis of the system. If the system has only one limit cycle, determining the position of the limit cycle is more important. In this paper, we construct an annular region containing all the limit cycles for a Kolmogorov-type system. Since the system contains the Lotka-Volterra, Gausse, Kuang and Freedman, and Huang models as special cases, the results obtained here are also valid for those models. Furthermore, the approach to the location of limit cycles in this paper is very applicable because the region is easy to compute.

Our Kolmogorov-type system is

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\phi(x)(F(x)-\pi(y)) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\rho(y)(\psi(x)-\xi(y)) \tag{1}$$

where x is the prey density, y is the predator density, $\phi(x)F(x)$ is the intrinsic growth rate of the prey in the absence of predators, and $\rho(y)\xi(y)$ is the intrinsic rate of the increase (or decrease) of the predator. The term $\phi(x)\pi(x)$ represents the functional response of the predator, i.e.

$$\phi(x)\pi(y)/x$$

is the rate of prey consumption per predator. Most authors simply take $\pi(y)=y$, but a function $\pi(y)$ that increases in a slow linear fashion could be used to model interference among the predators with each other's hunting, or a faster linear increase could be used to model predator cooperation (Harrison 1979). The term $\rho(y)\psi(y)$ is the response of the predator which means the difference of the actual rate of increase and the intrinsic rate of increase of the predator. The equation

$$(\rho(y)\psi(0)+\xi(y))/y$$

is the death rate of the predator in the absence of prey.

Clearly, system (1) contains the Lotka-Voleterra model, the Gause model, and the following models:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=xg(x)-\xi(y)p(x) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\zeta(y)(-\gamma+q(x)) \tag{2}$$

(Kuang and Freedman 1988) and

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\phi(x)(F(x)-\pi(y)) \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=\rho(y)\psi(x) \tag{3}$$

(Huang 1988a, 1989, Huang and Merill 1989).

The stability and instability of equilibrium points, the existence, uniqueness and non-uniqueness of limit cycles for the systems (3) have been thoroughly studied (Huang 1988a, b, 1989, Huang and Merill 1989). Of course, theorems for system (3) are also true for the Lotka–Volterra, Gause and Kuang–Freedman models because (3) is a generalisation of all these models.

For the Kolmogorov-type model (1), we have already proved the existence and uniqueness of limit cycles (Huang 1988c). In this paper, we try to estimate the relative position of the limit cycles.

Our fundamental assumptions in this discussion are the following.

(i) ϕ, ψ, π, ρ, $\xi \in C'[0,\infty)$; $\phi(0)=\pi(0)=\rho(0)=\xi(0)=0$; $\phi'>0$ for $x\geqslant 0$; $\pi'>0$, $\rho'>0$, $\xi'\leqslant 0$ for $y\geqslant 0$; there exist $\underline{x}>0$ such that $\psi(\underline{x})=0$ and $\psi'(x)>0$ for $x\neq\underline{x}$.

(ii) There exists $k>\underline{x}$ such that $F(k)=0$, $F'(k)<0$; $F(x)>0$ for all $0<x<k$, and for any $\bar{k}>k$, $F'(\bar{k})\neq 0$ if $F(\bar{k})=0$.

(iii) There exist positive numbers M and ε_M such that $\pi(y)\geqslant M\rho(y)$ for $y\geqslant\varepsilon_M$ and there also exist positive N and ε_N such that $\rho(y)>Ny$ for $y\geqslant\varepsilon_N$.

We restrict our discussion to the interior of the first quadrant Ω. These assumptions guarantee that there exist limit cycles surrounding the unique positive equilibrium (x^*,y^*), where $0<\underline{x}\leqslant x^*<k$, $y^*>0$. For the uniqueness of limit cycles we need one more assumption (Huang 1988c).

It is possible to have $F(0)=\infty$ in most of our discussion. In that case $(0,0)$ is no longer an equilibrium.

The problem of determining the relative positions of limit cycles is important especially when the system has only one limit cycle. Usually, authors use the Poincare–Bendixson annular boundaries to estimate the location of the limit cycles. (See, for example, Ye *et al* 1986). For predator–prey systems, several results have been reported (Conway and Smoller 1986, Freedman and Wolkowicz 1986, Kuang 1988a) but none of them are being carefully studied.

The following lemma is important for the proof of theorem 1.

Lemma 1. Every solution in Ω of the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\phi(x)(F(x^*)-\pi(y))\qquad\frac{\mathrm{d}y}{\mathrm{d}t}=\rho(y)(\psi(x)+\xi(y^*))\tag{4}$$

is periodic, where $\phi(x)$, $\psi(x)$, $\pi(y)$ and $\rho(y)$ satisfy the assumption (i).

Proof. Let $(x_0,y_0)\neq(x^*,y^*)$ be an initial point in Ω. The corresponding trajectory Γ of (4) satisfies

$$\int_{y_0}^{y}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y=\int_{x_0}^{x}\frac{\psi(x)+\xi(y^*)}{\phi(x)}\,\mathrm{d}x.\tag{5}$$

Let $\Gamma=(x(t),y(t))$. Suppose it is not a closed orbit. We can find two points $(x(t_1),y(t_1))$, $(x(t_2),y(t_2))$ with $t_1<t_2$ such that $x(t_1)=x(t_2)=x^*$, $y(t_1)$, $y(t_2)<y^*$.

Suppose, without loss of generality, $y(t_1)<y(t_2)$. Then

$$\int_{y_0}^{y(t_2)}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y=\int_{y_0}^{y(t_1)}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y+\int_{y(t_1)}^{y(t_2)}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y.$$

Since $F(x^*)-\pi(y)>0$ for $y<y^*$,

$$\int_{y_0}^{y(t_2)}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y>\int_{y_0}^{y(t_1)}\frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y.\tag{6}$$

But

$$\int_{x_1}^{x^*} \frac{\psi(x)+\xi(y^*)}{\phi(x)}\,\mathrm{d}x = \int_{y_0}^{y(t_1)} \frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y < \int_{y_0}^{y(t_2)} \frac{F(x^*)-\pi(y)}{\rho(y)}\,\mathrm{d}y = \int_{x_0}^{x^*} \frac{\psi(x)+\xi(y^*)}{\phi(x)}\,\mathrm{d}x. \qquad (7)$$

This contradiction completes the proof of lemma 1.

Theorem 1. Let $\bar{x}=\min\{x\,|\,F(x)=F(x^*)$ and $x>x^*\}$, and let $A=\{(x,y)\,|\,x^*\leqslant x\leqslant \bar{x},\ \pi(y^*)\leqslant\pi(y)\leqslant F(x)\}$. If

$$(F(x)-\pi(y))(\psi(x)+\xi(y^*))\geqslant(F(x^*)-\pi(y))(\psi(x)+\xi(y)) \qquad (8)$$

for $0<x<x^*$, then A is inside of all the limit cycles of (1).

Proof. Suppose L is a limit cycle of (1) surrounding (x^*, y^*). By the phase portrait analysis (Huang 1988c), L intersects the prey isocline $F(x)-\pi(y)=0$ exactly at two points.

Consider the system

$$\frac{\mathrm{d}x}{\mathrm{d}t}=\phi(x)(F(x^*)-\pi(y))$$

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\rho(y)(\psi(x)+\xi(y^*)) \qquad (9)$$

$$x(0)=x_0 \qquad y(0)=y_0.$$

Lemma 1 implies that all the solutions of (8) are periodic. Furthermore, each periodic orbit has two intersection points with the predator isocline $\psi(x)+\xi(y)=0$. Denote them as $P_0(x_0, y_0)$ and $S(P_0)=P_s(x_s, y_s)$. Clearly, $S(P_s)=S(S(P_0))=P_0$. Let $r(p)$ be the length of the curve along $\psi(x)+\xi(y)=0$ from points $(\underline{x}, 0)$ to P. Then the bigger the $r(P_0)$, the smaller the $r(P_s)$ (see figure 1).

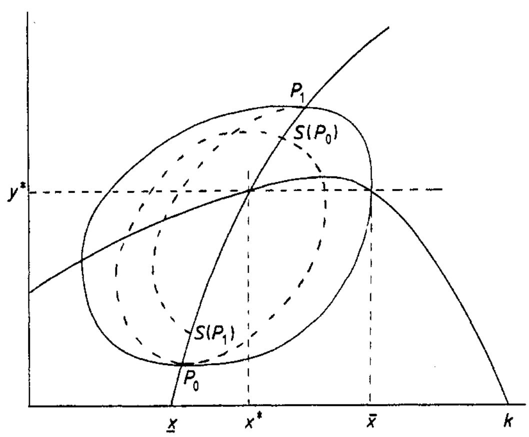

Figure 1. The flow of (1) (full curve) is always directed outwards with respect to the flow of (9) (broken curve).

According to the definition of A, if $(\bar{x}, y^*)$ is inside L, then so is A. Suppose $(\bar{x}, y^*)$ is not in L. Consider two vectors in the space

$$\begin{aligned} \bar{V}_1 &= (\phi(x)(F(x)-\pi(y)), \rho(y)(\psi(x)+\xi(y)), 0) \\ \bar{V}_2 &= (\phi(x)(F(x^*)-\pi(y)), \rho(y)(\psi(x)+\xi(y^*)), 0) \end{aligned} \tag{10}$$

and their vector product

$$\begin{aligned} \bar{V}_1 \times \bar{V}_2 = (0, 0, \phi(x)\rho(y)[(F(x)-\pi(y))(\psi(x)+\xi(y^*)) \\ -(F(x^*)-\pi(y))(\psi(x)+\xi(y))]). \end{aligned} \tag{11}$$

Since

$$\begin{aligned} &F(x)-\pi(y) > F(x^*)-\pi(y) && \text{for } x^* < x < \bar{x} \\ &\psi(x)+\xi(y^*) \geqslant \psi(x)+\xi(y) > 0 && \text{for } y \geqslant y^* \text{ and } (x, y) \in A \end{aligned}$$

then

$$(F(x)-\pi(y))(\psi(x)+\xi(y^*)) \geqslant (F(x^*)-\pi(y))(\psi(x)+\xi(y)) \tag{12}$$

for $x^* < x < \bar{x}$. Therefore, from (8) and (12) we have, for $0 < x < \bar{x}$,

$$\phi(x)\rho(y)[(F(x)-\pi(y))(\psi(x)+\xi(y^*))-(F(x^*)-\pi(y))(\psi(x)+\xi(y))] \geqslant 0.$$

Hence, the flow of (1) is always directed outwards with respect to the flow of (9).

Thus, if $P_0(x_0, y_0)$, $P_1(x_1, y_1)$ are on L and $P_0(x_0, y_0)$, $P_s(x_s, y_s)$ are on one of the closed orbits of (9), then, comparing the trajectories of the two closed orbits in $\Omega_1 = \{(x, y) \mid \psi(x)+\xi(y) > 0\}$, $r(P_1) > r(P_s)$ (see figure 1).

Similarly, in $\Omega_0 = \{(x, y) \mid \psi(x)+\xi(y) < 0\}$, considering the trajectories of two closed orbits initiating at $P_1(x_1, y_1)$, we have $r(P_0) < r(S(P_1))$. Since $r(P_1) > r(P_s)$, $r(S(P_1)) < r(S(P_s)) = r(P_0)$. This is a designed contradiction which ends the proof of theorem 1. □

Theorem 2. Let $l > 0$ be a constant such that

$$l\frac{\mathrm{d}x}{\mathrm{d}t} + \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{y=l(k-x)} \leqslant 0. \tag{13}$$

Then all the limit cycles of the system (1) are contained in the region B, where $B = B_1 \cup B_2$,

$$\begin{aligned} B_1 &= \{(x, y) \mid 0 \leqslant x \leqslant x^*, 0 \leqslant y \leqslant l(k-x^*)\} \\ B_2 &= \{(x, y) \mid x^* \leqslant x \leqslant k, 0 \leqslant y \leqslant l(k-x)\}. \end{aligned}$$

Proof. Define vectors $\bar{V}$ and $\bar{T}$ as

$$\bar{V} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, 0\right)$$

$$\bar{T} = (t_1, t_2, t_3) = \begin{cases} (-1, 0, 0) & \text{if } 0 \leqslant x \leqslant x^*, y = l(k-x^*) \\ (-1, l, 0) & \text{if } x^* \leqslant x \leqslant k, y = l(k-x). \end{cases} \tag{14}$$

Since

$$\bar{T} \times \bar{V} = \left(0, 0, -\left(\frac{\mathrm{d}y}{\mathrm{d}t} + t_2\frac{\mathrm{d}x}{\mathrm{d}t}\right)\right)$$

then if we can prove that

$$\frac{dy}{dt}+t_2\frac{dx}{dt}\leqslant 0 \qquad \text{for } 0\leqslant x\leqslant k$$

then B is invariant under (1).

By (14), $t_2=0$ when $0\leqslant x\leqslant x^*$, and then

$$\frac{dy}{dt}+t_2\frac{dx}{dt}=\rho(y)(\psi(x)+\xi(y))\leqslant 0$$

$t_2=l$, $y=l(k-x)$ when $x^*\leqslant x\leqslant k$, and consequently $dy/dt+l\,dx/dt\leqslant 0$ by (13). □

Therefore, B contains all the limit cycles of the systems (1) since B is invariant under (1).

Theorems 1 and 2 imply the following estimation of the relative position of the limit cycles of (1).

Theorem 3. In addition to the assumptions (i)–(iii), if (8) and (13) hold, then all the limit cycles of (1) are in the annular region $B\backslash A$.

Let us conclude by discussing a number of points.

(1) The sets A and B are easily constructed and the region is explicitly computable. Thus, the theorems here are practically useful.

(2) It is easy to see that the set A can be extended as

$$A'=\{(x,y)\,|\,x^*\leqslant x\leqslant \bar{x},\ \psi(x)+\xi(y)>0,\ y^*\leqslant y\leqslant \pi^{-1}(F(x_M))$$
$$\pi(y)\leqslant F(x) \text{ for } x_M\leqslant x\leqslant \bar{x} \qquad \text{where } F(x_M)=\max_{x^*\leqslant x\leqslant\bar{x}}\{F(x)\}\}.$$

(3) If $\xi(y)=0$ in (1), then the system reduces to the system (3) (Huang 1988a, b, 1989, Huang and Merrill 1989). The condition (8) in Theorem 1 can be reduced to

$$F(x)\leqslant F(x^*) \qquad \text{for } 0<x<x^*$$

which is easier to check.

(4) The technique in this paper is similar to that used by Kuang (1988), but his model is a special case of our system. Also, he needs more assumptions for the existence of limit cycles in his model and his proof needs to be amended. For example, he used the expression $xg(x)/p(x)$ very often, such as

$$a_1=\max\{\xi^{-1}(xg(x)/p(x))\,|\,x\in(0,k)\} \qquad a_2=\min\{\xi^{-1}(xg(x)/p(x))\,|\,x\in(0,x^*)\}$$

(see p 78 of Kuang 1988), but it is not always valid since $p(0)=0$.

(5) We can use the same technique to obtain better estimates of the relative position of limit cycles, but, of course, it needs more computation.

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